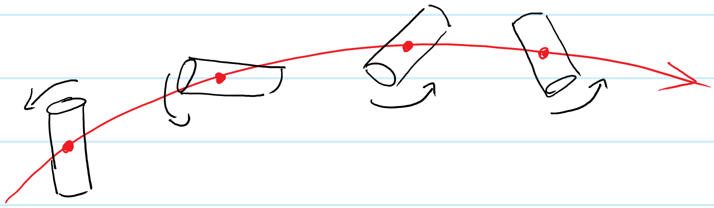
Lecture 8 Dynamics of Rigid Body I

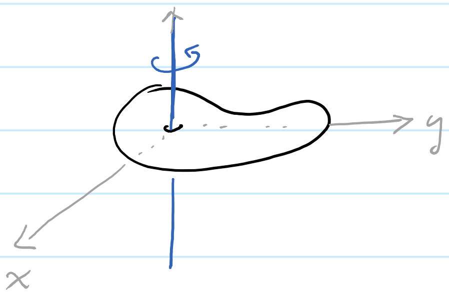
$\vec{F}_{net\,ext} = m\vec{a}_{cm}$ deals with trajectory of CM



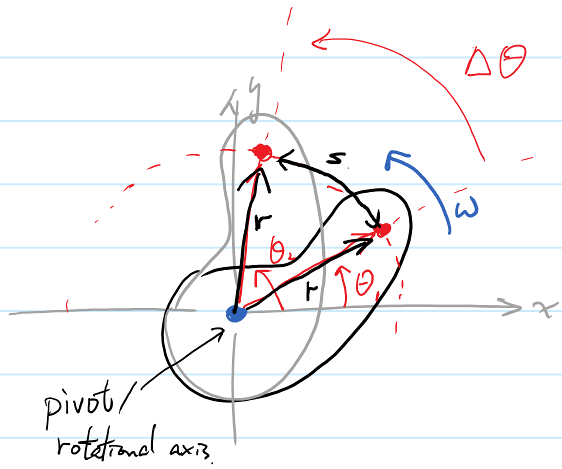
but not rotation of the object

- Rigid Body : No deformation allowed (stretching, twisting)

Rotational Kinematics



Top view $\longrightarrow$



Angular displacement $= \Delta\theta = \theta_2 - \theta_1 = \dfrac{s}{r}$ in radian. (rad)

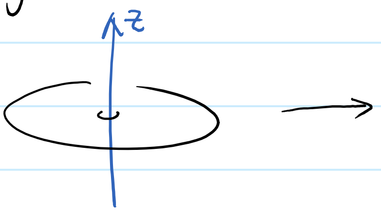
Angular velocity $= \omega = \dfrac{\Delta\theta}{\Delta t} \longrightarrow \omega = \dfrac{d\theta}{dt}$ in rad/s

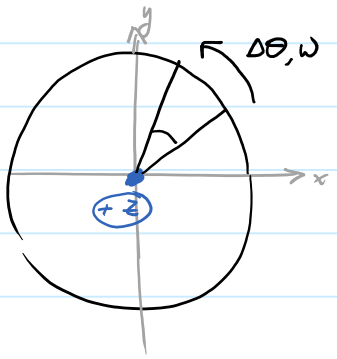
Angular acceleration $= \alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$ in rad/s²

- cf. linear kinematics $(\Delta x,\ v,\ a)$

Note: $\Delta\theta$, ω and α are the same for all particles in the same body.

Sign Convention



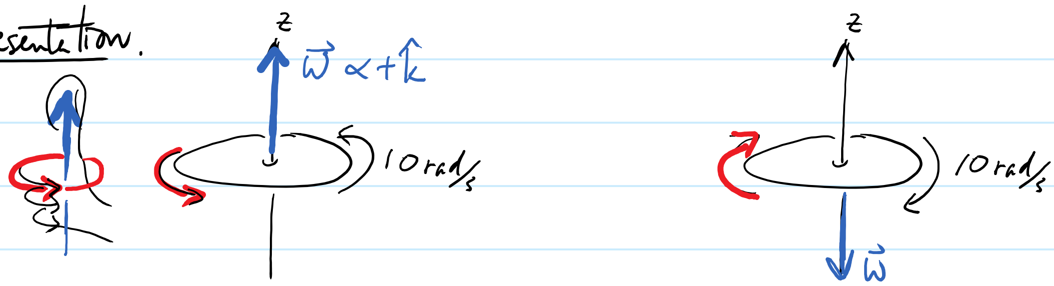


$\Delta\theta > 0$ $\qquad$ $\Delta\theta < 0$

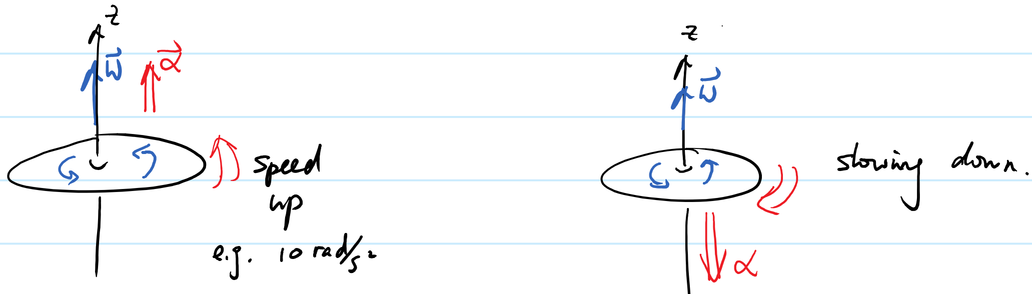
$\omega_z > 0$ $\qquad$ $\omega_z < 0$

Right-hand-rule . .

Vector Representation.

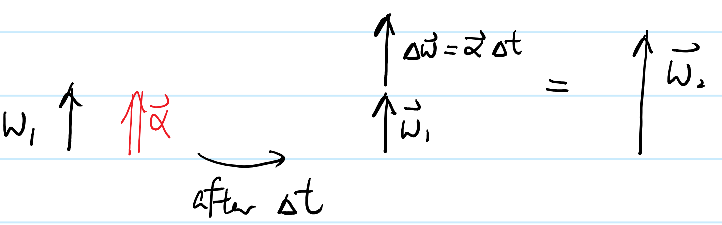


vector $\longrightarrow$ $\vec{\omega} = +(10\ \mathrm{rad/s})\ \hat{k}$ $\qquad$ $\vec{\omega} = -10\ \mathrm{rad/s}\ \hat{k}$.

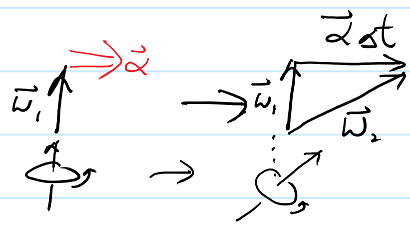


$\vec{\alpha} = +10\ \mathrm{rad/s^2}\ \hat{k}$ $\qquad$ $\vec{\alpha} = -10\ \mathrm{rad/s^2}\ \hat{k}$

Note: $\quad \vec{\alpha} \sim \dfrac{\Delta\vec{\omega}}{\Delta t} \Rightarrow \Delta\vec{\omega} = \vec{\alpha}\,\Delta t$



Can $\vec{\alpha}$ change direction of $\vec{\omega}$? Yes.



Fixed axis rotation

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\omega = \frac{d\theta}{dt}$$

$$\Rightarrow \begin{cases} \omega(t) = \omega(t_0) + \int_{t_0}^{t} \alpha(t')\,dt' \\ \theta(t) = \theta(t_0) + \int_{t_0}^{t} \omega(t')\,dt' \end{cases}$$

if α is constant, take $t_0 = 0$

$$\Rightarrow \begin{cases} \omega_f = \omega_i + \alpha t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \end{cases}$$

$v = u + at$

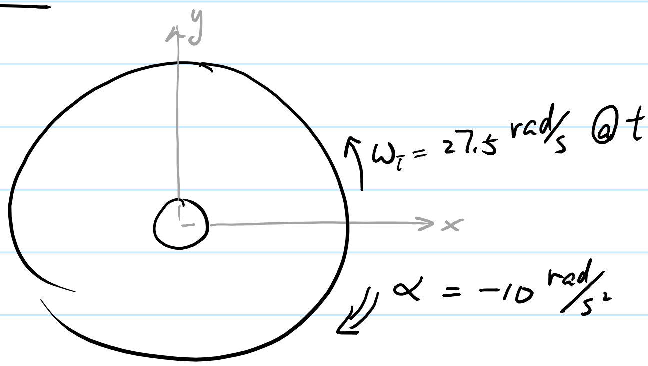
$x = x_0 + v_0 t + \frac{1}{2}at^2$

same as relation as in linear kinematic quantities

i.e. $\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$, sign of α and $\Delta\theta$ follows the convention.

cf. $v^2 = u^2 + 2as$

Example: DVD



Question:

① How long will it take to stop?

② What is the total angular displacement?

ANS.

①
$\omega_f = \omega_i + \alpha t$, $\omega_f = 0$, $\omega_i = 27.5\ \text{rad/s}$, $\alpha = -10\ \text{rad/s}^2$

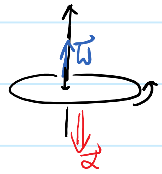
$$\Rightarrow t = \frac{27.5}{10} = 2.75\ \text{s}$$

②
$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \Rightarrow \Delta\theta = \theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$$

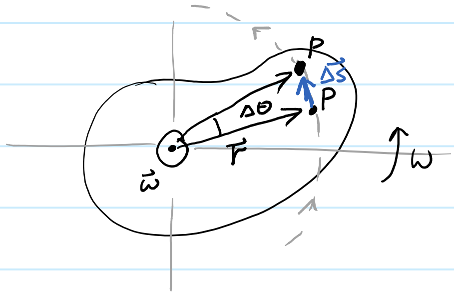
$$\Delta\theta = 27.5(2.75) + \frac{1}{2}(-10)(2.75)^2 = 37.8\ \text{rad} = 6.02\ \text{revolutions.}$$

$37.8/2\pi = 6.02$.

What is the direction of $\vec{\omega}$ and $\vec{\alpha}$ in the DVD example?



Relation to linear velocity and acceleration.



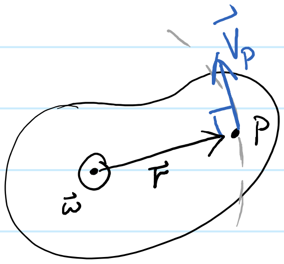
any point P moves in a circular motion with radius r.

in a time interval, Δt.

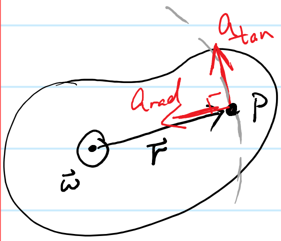
Distance travelled by P $= |\Delta \vec{s}| = \Delta s$

angular displacement $= \Delta\theta$

Since $r\Delta\theta = \Delta s \rightarrow ds = r\,d\theta$

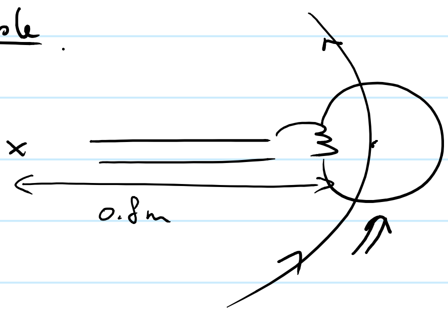


$$\vec{V}_P \begin{cases} V_{tan} = \dfrac{ds}{dt} = r\dfrac{d\theta}{dt} = r\omega \\ V_{rad} = 0 \quad \because r \text{ of } P \text{ does not change} \end{cases}$$



$$\left.\begin{cases} a_{tan} = \dfrac{dV_{tan}}{dt} = \dfrac{d(r\omega)}{dt} = r\dfrac{d\omega}{dt} = r\alpha \\ a_{rad} = a_{centripetal} = \dfrac{V^2}{r} = \dfrac{V_{tan}^2}{r} \end{cases}\right\} \text{both non-zero}$$

Example.



$\omega = 10 \text{ rad s}^{-1}$

$\alpha = 50 \text{ rad s}^{-2}$

$a_t = r\alpha = 0.8 \cdot 50 = 40 \text{ m/s}^2$

$a_r = \dfrac{V^2}{r} = \dfrac{(r\omega)^2}{r} = r\omega^2$

$= 0.8 \times (10)^2 = 80 \text{ m/s}^2$

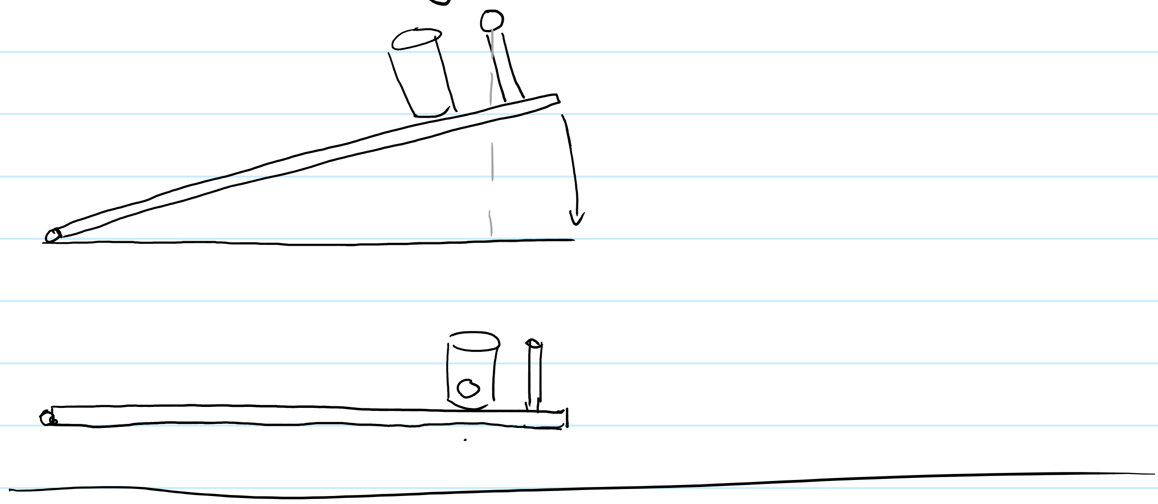
$|\vec{a}| = \sqrt{a_t^2 + a_r^2} = 89.4 \text{ m/s}^2$

Force applying on the disc.

$|\vec{F}| = m_{disk} \cdot |\vec{a}| = 143.04\text{N}.$

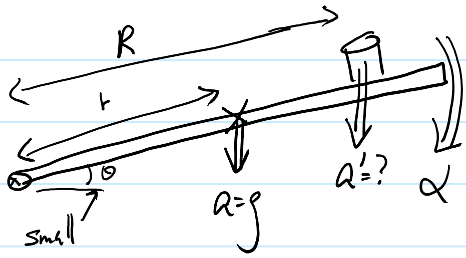
$\uparrow 1.6 \text{ kg}$

Demo   faster than g



The ball falls at $a = g = 9.8\ m/s^2$.

But the cup has a larger linear acceleration than $9.8\ m/s^2$. Therefore, it reach the ground before the ball, allowing the ball to fall into it.



There are forces acting on the ramp at the joint (pivot). But the vertical force is small and can be ignored.

Proof

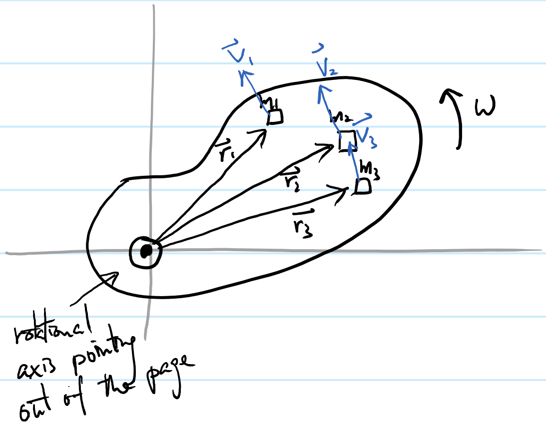
The only vertical net force is gravity. For any system, the CM accelerates according to the net external force only. $\Rightarrow \vec{a}_{cm} = \frac{\vec{F}_{ext}}{M_{tot}} = \frac{M_{tot}\vec{g}}{M_{tot}} = \vec{g}$.

The cup is further away from the pivot, at distance R. But both CM and the cup are also rotating about the pivot. Assume the angle is small,

$$\frac{a_{cm}}{r} = \alpha = \frac{a_{cup}}{R} \quad (\because \alpha \text{ is the same for entire rigid body.})$$

$$\Rightarrow a_{cup} = a_{cm} \cdot \frac{R}{r} > a_{cm} = g$$

Kinetic energy in rotation. (fixed axis)



$$K = \sum_i \frac{1}{2} m_i v_i^2 \qquad v_i = r_i \omega$$

$$= \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$= \frac{1}{2} \underbrace{\left(\sum_i m_i r_i^2\right)}_{I} \omega^2$$

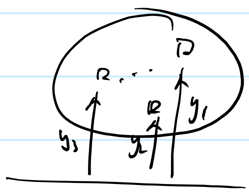
r_i are perpendicular distance from the rotational axis to the mass m_i

$$\Rightarrow K = \frac{1}{2} I \omega^2 \quad \text{where} \quad I = \sum m_i r_{i\perp}^2 = \text{moment of inertia.}$$

cf. $K = \frac{1}{2} m v^2$ $\quad \left\{ \begin{matrix} m & \leftrightarrow & I \\ v & \leftrightarrow & \omega \end{matrix} \right\}.$

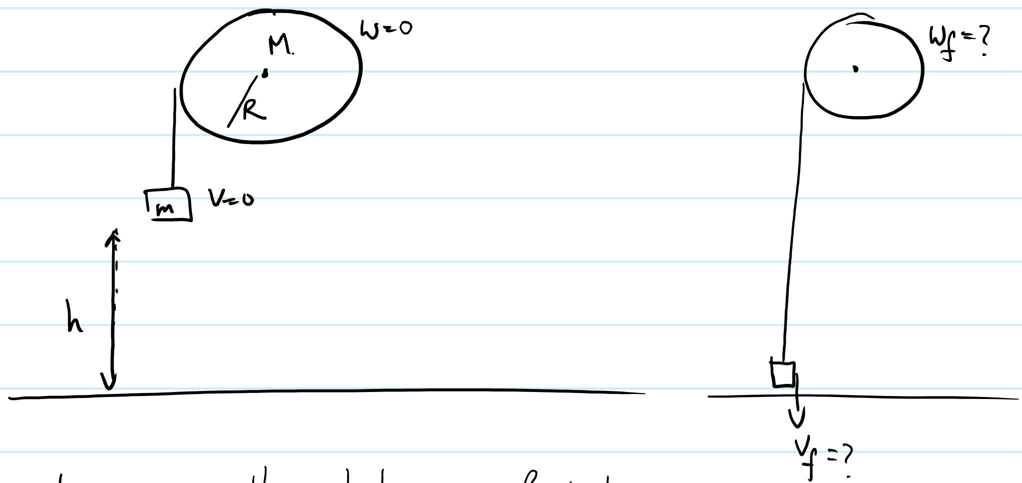
Gravitational potential energy of rigid body $= \sum_i m_i g y_i = \underbrace{\sum_i m_i y_i}_{} \cdot g$

$$= m_{tot} \, y_{cm} \cdot g.$$



$$\Rightarrow \boxed{U_g = m_{tot}\, g\, y_{cm}.}$$

Example.



Assumptions: ① The wheel is frictionless.

② No slipping between the cylinder and the cable.

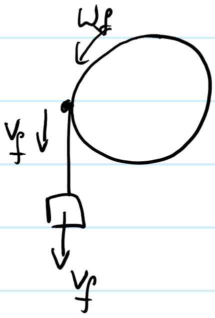
$W_{N.C.} = 0 \quad \Rightarrow \quad \Delta K + \Delta U = 0$

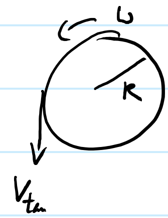
$$E_{tot} = K_m + K_{wheel} + U_m + U_{wheel}$$

$$\Delta K_m = \frac{1}{2} m v_f^2 - 0 \qquad = \frac{1}{2} m v_f^2$$
$$\Delta K_{wheel} = \frac{1}{2} I \omega_f^2 - 0 \qquad = \frac{1}{2} I \omega_f^2 \qquad\qquad I = \frac{1}{2} M R^2$$
$$\Delta U_m = 0 - mgh \qquad = -mgh$$
$$\Delta U_{wheel} = 0 \qquad = 0$$

$$\Rightarrow \quad \frac{1}{2} m v_f^2 + \frac{1}{2} I \omega_f^2 - mgh = 0$$

2 unknowns!





$v_{tan} = R\omega$

$\Rightarrow v_f = R\omega_f$ or $\omega_f = \frac{v_f}{R}$.

$$\Rightarrow \quad \frac{1}{2} m v_f^2 + \frac{1}{2} I \frac{v_f^2}{R^2} = mgh$$

$$\Rightarrow \quad v_f = \sqrt{\frac{2mgh}{m + I/R^2}} < \sqrt{2gh}$$

free fall situation

Parts of the initial potential energy of the mass, m, convert to the kinetic energy of the rotating wheel. Therefore, the kinetic energy of the mass becomes smaller when compare to the case of free falling.